

A. Cintio

Dipartimento di Fisica and INFN, Pisa, Italy

G. Morchio

Dipartimento di Fisica and INFN, Pisa, Italy

Sum rules and density waves spectrum for non relativistic fermions

Abstract

Frequency sum rules are derived in extended quantum systems of non relativistic fermions from a minimal set of assumptions on dynamics in infinite volume, for ground and thermal states invariant under space translations or a lattice subgroup.

For the jellium Coulomb model, they imply the one point result for the plasmon energy spectrum in the zero momentum limit.

In general, the density waves energy spectrum is shown to converge, in the limit of large wavelength, to a point measure at zero frequency, for any number of fermion fields and potentials $V^{(ij)}$ with integrable second derivatives.

For low momentum, $\langle \omega^2(k) \rangle \sim k^2$ for potentials with $r^2 \partial_i \partial_j V$ integrable, $\langle \omega^2(k) \rangle \sim k^{\alpha-d+2}$ for potentials decaying at infinity as $1/r^\alpha$, $d-2 < \alpha < d$, d the space dimensions.

For one component models with short range interactions, the fourth momentum of the frequency is expressed, at lowest order in k , purely in terms of the three point correlation function of the density.

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1. Introduction

The physics of charge density waves in extended systems was understood in ref.[1] in terms of classical charge configurations giving rise to slowly varying electric fields, resulting in *plasma oscillations* with frequency $\omega_p^2 = e^2 \rho / m$ for infinitely extended systems of particles of mass m , charge e and density ρ .

In solid state physics such an analysis applies [2] on the basis of suitable simplifications, in particular of a *random phase approximation* [3] in the analysis of charge density correlation functions. Plasma oscillations correspond to a single point frequency spectrum, $\omega = \omega_p$, for the charge density correlation function at zero momentum, and it is not clear whether neglected terms may spoil such pure point result, even in the case of only one kind of charged particles in a neutralizing uniform background (jellium model).

The control of the energy spectrum associated to density waves also plays a crucial rôle in the theory of quantum liquids, in particular in the discussion of superfluidity, [4] [5] [6]; very similar problems appear in the discussion of the current commutators at low momentum which are at the basis of the Kubo and Landau approaches to superconductivity [7] [8].

In ref. [9] the relevant spectrum in the jellium Coulomb model was derived, and shown to consist only of the plasma frequency, through an analysis of Galilei transformations, which are spontaneously broken in the jellium model, as in any nonrelativistic system at non-zero density.

The derivation employed a *generalized Goldstone theorem*, which gives the energy spectrum associated to spontaneously broken symmetries in terms of commutators between generators of symmetries and order parameters. Such commutators are time independent in the ordinary Goldstone case, but not in the presence of sufficiently long range interactions, the Coulomb potential in the jellium case. Similar results have also been obtained in refs. [10] through an analysis of operators describing long wave charge fluctuations in the jellium and other models.

The Galilei analysis is complicated by the lack of invariance under space translations of the density of the Galilei generator and by short distance problems associated to the singularity of the Coulomb potential at the origin. The analysis can in fact be simplified by focusing on charge density commutators, a technique discussed e.g. in refs. [11], [12], [13]. On one side, such commutators provide an alternative derivation of the spectrum associated to Galilei transformations, on the other they have convenient positivity properties and simple evo-

lution equations. Moreover, as we shall see, the charge commutators analysis applies to a large class of systems and allows for the derivation of exact relations for (density waves) energy spectra, also at non-zero momentum.

The aim of this paper is to present a self contained derivation of the plasma spectrum at zero momentum in the jellium model, based on the analysis of charge commutators, to extend the analysis to the case of spontaneous breaking of translation to a lattice subgroup (which is relevant in presence of Wigner crystallization [14], [15]) and to derive, by the same methods, general relations for the frequency spectrum at low momenta of density waves in non relativistic fermion systems with short and long range potentials.

The use of sum rules to obtain information on the density waves frequency spectrum seems to have been prevented by results on the divergence of momenta of order ≥ 3 [16]. However, the divergence of the third momentum only appears for singular (delta or hard core) potentials and the divergence of the fifth momentum in the Coulomb case depends on the singularity at the origin of the potential. More generally, the (perturbative) results of [16] indicate that no momentum diverges for regular (C^∞) potentials, a result which also follows from our analysis. In the following, the Coulomb potential will be regularized at the origin and the limit $k \rightarrow 0$ of the energy spectral measure will be shown to be independent of the regularization.

We remark that the result on the plasma spectrum at zero momentum does not follow from the sum rules for the frequency momenta up to the third, nor by the use of the “perfect screening sum rule”, concerning $\langle \omega^{-1} \rangle$ [6], unless a one point approximation is assumed for the frequency spectrum; however, the reduction of the spectrum to a single point is the key result, which involves the fifth momentum and holds, in our analysis, only in the limit $k \rightarrow 0$.

Besides the analysis of the plasma spectrum and of the fifth frequency momentum, we reobtain the third momentum sum rules of refs.[11] [17] [12], clarifying and symplifying their derivation, extending them to states with discrete translation symmetry, and deriving exact consequences on energy spectra, in particular in the case of long range potentials with a faster decay with respect to the Coulomb potential. Our results do not require the explicit construction of ground and thermal states and may also shed light on the problems and alternatives which appear in their analysis [18], [19], [20], [21].

We consider infinite systems of nonrelativistic fermions described by the canonical anticommutation relations (CAR) algebra, in d space dimensions, in particular $d = 2, 3$. The time evolution is assumed to be given, through equal time commutators, by a free Hamiltonian and

interaction potentials $V(|x|)$. We consider states Ω_β , with nonzero mean particle density $\langle \rho \rangle$, invariant under space translations or under a lattice subgroup of them, time evolution and parity, satisfying either the spectral condition, i.e. positivity of the energy in the resulting representation of the CAR algebra, or the KMS condition at inverse temperature β .

We denote by $\mathcal{F}(f)$ or by \tilde{f} the Fourier transform of f , by $f * g$ the convolution of f and g , by $A(f)$ the smearing of the distribution $A(x)$ with f ; unless specified differently, test functions belong to the Schwartz space \mathcal{S} of infinitely differentiable functions of fast decrease. For simplicity, the Planck constant (divided by 2π) is omitted and the variable ω is used for the spectrum of the Hamiltonian; sum over repeated indexes is implicit and dx will denote $d^d x$. Our results are the following:

i) one-point plasma spectrum for jellium at zero momentum:

In the jellium Coulomb model, i.e. for non-relativistic fermions in three space dimensions with interaction $V(x) = e^2/4\pi|x| * \gamma(|x|)$, $\gamma \in \mathcal{S}$ real, $\int \gamma(x) dx = 1$, we consider the expectations

$$d\mu_f(\omega) = \int \langle \rho(\tilde{f}) dE(\omega) \rho(f) \rangle \ ,$$

with $f(x) = \exp ikx \alpha_R(x)$ (see eq.(7), $dE(\omega)$ the spectral measure of the Hamiltonian, $\langle \rangle$ the expectation on a thermal or ground state, invariant under space translations or a lattice subgroup of them and parity. The normalized positive measures

$$d\nu_f(\omega) \equiv N_f \omega (d\mu_f(\omega) - d\mu_f(-\omega)) \ .$$

converge, for $\alpha_R \rightarrow 1$ and then $k \rightarrow 0$, to

$$1/2 (\delta(\omega - \omega_p) + \delta(\omega + \omega_p)) \ , \quad \omega_p^2 = e^2 \langle \rho \rangle / m \quad (1)$$

independently of the regularization γ . The derivation relies purely on Newton equations, Gauss' law at large distances and invariance under (a lattice subgroup of) translations.

ii) Goldstone spectrum for potentials with integrable second derivatives:

For any number of fermion fields and all regular potentials $V^{(kl)}$ with $\partial_i \partial_j V^{(kl)}(x)$ integrable, the above measures $d\nu_f$, defined by the total density $\rho = \sum_i \rho_i$, converge to $\delta(\omega)$, corresponding to a *Goldstone spectrum* (also associated to the spontaneous breaking of the Galilei charges $\sum_i \int dx \rho_i(x) x_k$).

iii) dependence of the spectrum on the decay properties of the potential:

In the same cases as in ii) and with the same notation, for $R \rightarrow \infty$, i.e $f \rightarrow \exp ikx$,

$$\int \omega^2 d\nu_f(\omega) \rightarrow c |k|^2 (1 + o(|k|))$$

if $x^2 \partial_i \partial_j V(x)$ is integrable;

$$0 \int \omega^2 d\nu_f(\omega) \leq c |k|^\beta$$

if $|x|^\beta \partial_i \partial_j V(x)$, $0 < \beta < 2$, is integrable, with an equality, up to $o(|k|)$, for one fermion field and $V(x) \sim 1/|x|^\alpha$, $\alpha = \beta + d - 2$. In particular, $\langle \omega^2(k) \rangle = O(|k|)$ for jellium Coulomb systems (i.e. $V \sim 1/|x|$) in two space dimensions.

iv) fourth momentum of the spectrum at the order k^2 :

For one fermion field, regular potentials of fast decrease and states invariant under space translation, rotations and parity the fourth momentum of $d\nu_f(\omega)$ converges, for $f \rightarrow \exp ikx$, to

$$k^2 / \langle \rho \rangle m^2 \int dx dy W(x, y) \langle \rho(0) \rho(x) \rho(y) \rangle + O(k^4) , \quad (2)$$

with $W(x, y)$ computed in Sect.6 in terms of the potential. In particular, $\langle \omega^4(k) \rangle$ is not of the same order as $\langle \omega^2(k) \rangle^2$, as it would follow from a single quasi-particle interpretation.

In Sect. 2 the mathematical framework is specified, in terms of correlation functions for infinite systems, with time derivatives given by appropriate commutators with local hamiltonians. Energy spectra are expressed in terms of time derivatives of commutators; convergence to delta functions of the spectral measures in the limit of zero momentum follows from relations between frequency momenta.

In Sect. 3 equations of motion, commutators and low momentum expansions are discussed in general, together with their implications on energy-momentum spectra for short range interactions.

In Sect. 4 the expression for $\langle \omega^2(k) \rangle$ is shown to result, for long range potentials, in the above dispersion relations.

In Sect. 5 the fourth momentum of the frequency is computed in the jellium model in the limit $k \rightarrow 0$, implying the plasma frequency result.

In Sect. 6 the first term in the k expansion of $\langle \omega^4(k) \rangle$ is expressed in terms of the three point function of the density for regular short range potentials.

2. Commutators and energy spectra

We consider infinite systems, described by fermion fields $\psi_i^*(f), \psi_i(f)$, generating an ACR algebra [22], with

$$[\psi_i^*(f), \psi_j(g)]_+ = \delta_{ij} \int f(x)g(x) dx \quad , \quad [\psi_i(f), \psi_j(g)]_+ = 0 \quad , \quad (3)$$

dx staying, here and in the following, for $d^d x$ in d space dimensions. The addition of spin indexes leads to minor changes, which will be indicated when relevant. The results of Sects. 3 and 4 hold for any number of fermion fields, those of Sects. 5 and 6 in the case of only one fermion field.

We consider representations defined (through the GNS construction [22], [23]) by states invariant under (a lattice subgroup of) space translations, with a finite number of particles when restricted to finite regions, i.e. defining locally Fock representations. This allows for the use of variables in the weak closure of the ACR algebra in the Fock space, e.g. bounded functions of the operators $\rho(f)$, (density operators integrated with regular functions), and in fact we will work, in the spirit of Wightman theory [24], with unbounded field operators like $\rho(f)$.

In the presence of interactions, the construction of the dynamics of such systems in terms of automorphisms of the ACR algebra is not completely under control; in the case of lattice spin systems [25], integrability of the interaction implies norm convergence of finite volume dynamics and stability of the quasi-local algebra, while the results of [26] and [9] imply the necessity of weaker convergence and larger algebras in the case of Coulomb interactions.

For our purposes, it is enough that dynamics exists as a group of automorphisms of an algebra containing the ACR algebra and that expectation values of time derivatives are given, at zero time and for suitable variables, by limits of commutators with hamiltonians associated to a sequence of bounded regions invading the space. We end therefore with the following assumptions, in the spirit of Wightman theory:

A) The correlation functions at all times of the fermion fields and of their Wick ordered polynomials are distributions in the space variables, continuous in time as distributions, invariant under time translations and space translations, or under a lattice subgroup of the latter, satisfying Wightman positivity and therefore defining operator valued distributions on the invariant Wightman domain, with fundamental vector Ψ_0 . The corresponding state (invariant under time and space, or lattice, translations) will be denoted by Ω , expectations on Ω by $\langle \rangle$

In the space translation invariant case, the unitary groups implementing space and time translations are then strongly continuous; their generators will be denoted by P and H , and the corresponding joint spectral measure by $dE(k, \omega)$. For states invariant under a lattice subgroup of space translations, $dE(k, \omega)$ will denote the spectral measure associated to time and lattice translations, with k in the fundamental cell of the reciprocal lattice; $dE(\omega)$ will denote the spectral measure of H .

B) Ω will be assumed to be a ground or a thermal state, i.e. to satisfy either positivity of the energy $H \geq 0$, or the KMS conditions

$$\Omega_\beta (A B(t)) = \Omega_\beta (B(t - i\beta) A) \quad (4)$$

with $B(t - i\beta)$ defined by analytic continuation of the correlation functions. The correlation functions at equal time will be assumed to be bounded, after smearing, with respect to translations of space variables, to satisfy the cluster property and to be invariant under parity. Invariance under rotations will be assumed only to simplify the results, in the space translation invariant case.

C) The correlation functions of time derivatives, at equal times, of Wick polynomials $\Phi(x)$ are given by the infinite volume limit of the correlation functions of their (multiple) commutators with the local hamiltonians

$$\begin{aligned} (-i)^n d^n/dt^n \Phi &= [H \dots [H, \Phi] \dots] = \\ &= \lim_{L_i \rightarrow \infty} \lim_{R_j \rightarrow \infty} [H_{L_1, R_1} [\dots [H_{L_n, R_n}, \Phi] \dots]] \end{aligned} \quad (5)$$

with

$$\begin{aligned} H_{L,R} &\equiv H_R^0 + H_R^\rho + H_{L,R}^{int} \equiv \\ &\equiv 1/2m \int dx \alpha_R(x) \partial_i \psi^*(x) \partial_i \psi(x) + \mu \int dx \alpha_R(x) \psi^*(x) \psi(x) + \\ &+ 1/2 \int dx dy \alpha_R(x) \alpha_R(y) \psi^*(y) \psi^*(x) V_L(x - y) \psi(x) \psi(y) \quad , \quad (6) \end{aligned}$$

with $V_L(x) = \alpha_L(x) \gamma * V(|x|)$, $\gamma(|x|) \in \mathcal{S}$ real, $\int \gamma d^d x = 1$, and

$$\alpha_R(x) \equiv \alpha(|x|/R) \quad , \quad \alpha \in \mathcal{S} \quad , \quad \alpha(x) \geq 0, \quad \alpha(0) = 1 \quad , \quad \alpha(x) = 0 \quad \forall x \geq 1 \quad (7)$$

and similarly for more than one fermion field; the limits $R_j \rightarrow \infty$ exist by locality of the ACR relations; existence, and independence of the order, of the limits $L_i \rightarrow \infty$ follows from integrability of $V(x)$ and boundedness of the correlation functions in the above sense.

For gauge invariant variables, the only ones to be considered in the following, the second term in eq. (6), as well as a possible term $\rho V_L * \langle \rho \rangle$, corresponding to an interaction with a uniform background, are irrelevant in the above commutators; moreover, the integrability condition applies to $\partial_i V(x)$, since only derivatives of V appear, as a consequence of the vanishing of the commutators between gauge invariant variables and the integral of the charge density.

For Coulomb interactions, in order to perform the limits $L_i \rightarrow \infty$, the truncated correlation functions will be assumed to decay, after smearing with test functions, as $\prod_{i=1}^{n-1} |x_{\sigma(i)} - x_{\sigma(i+1)}|^{-1-\varepsilon}$ for all permutations σ , for some $\varepsilon > 0$. In general, similar decay properties are required in the absence of integrability of $\partial_i V$. The ultraviolet regularization is necessary, in general, for the existence of frequency momenta. It will be omitted in the notation, writing V for $\gamma * V$.

In the above framework, we will derive constraints on energy spectra which arise directly from the equations of motion. Goldstone theorem can be regarded as one of them, and follows in fact [26] from invariance of the equation of motion under a symmetry which commutes with space translations and is spontaneously broken, under sufficient locality properties of the time evolution. For long range interactions, the latter property may fail, and in this case the proof of Goldstone theorem gives a relation between an energy spectrum at zero momentum and the time dependence of the expectation value of the commutator between an order parameter A and a charge operator Q_R , in the limit $R \rightarrow \infty$ [27] [26]

In this paper, similar information on energy spectra will be obtained from an analysis of time derivatives of commutators of the form $\langle [A^*(x), A(y, t)] \rangle$. If $A \Psi_0$ is in the domain of H^n , then

$$(-i)^n d^n / dt^n \langle A^* A(t) \rangle_{t=0} = \int \omega^n \langle A^* dE(\omega) A \rangle .$$

The energy spectral measure of the state $A \Psi_0$ will be denoted by $d\mu_A(\omega) \equiv \langle A^* dE(\omega) A \rangle$; the measures

$$d\nu_A(\omega) \equiv \omega (d\mu_A(\omega) - d\mu_{A^*}(-\omega)) \quad (8)$$

satisfy, under the above domain conditions,

$$\int d\nu_A(\omega) \omega^n = (-i)^{n+1} d^{n+1} / dt^{n+1} \langle [A^*, A(t)] \rangle_{t=0} , \quad (9)$$

For ground and thermal states the measures $d\nu_A$ are positive. If $d\mu_{A^*}(\omega) = d\mu_A(\omega)$, which will follow in our case from parity invariance, they are even and determine $d\mu_A$ up to $\delta(\omega)$:

Lemma 1 *If Ω is a ground or KMS state and A hermitean, $d\nu_A(\omega)$, defined by eq.(8), is positive. If $d\mu_{A^*} = d\mu_A$, then $d\nu_A$ is even and determines $d\mu_A$ apart from multiples of $\delta(\omega)$; in this case, if $d\nu_{A_n}$ converge as measures, $d\mu_{A_n}$ converge as distributions, apart from $\delta(\omega)$ terms.*

Proof: For ground states, $d\mu(\omega)$ has positive support, so that $d\nu$ is clearly positive and determined up to $c\delta$. For KMS states, $d\mu(-\omega) = \exp(-\beta\omega) d\mu(\omega)$, so that

$$d\nu(\omega) = \omega (1 - e^{-\beta\omega}) d\mu(\omega) ,$$

which implies positivity of $d\nu$ and uniqueness of $d\mu$ apart from multiples of $\delta(\omega)$. Convergence in the sense of distributions of the odd part of $d\mu_{A_n}$ follows from eq.(8) and implies, by the ground state or the KMS condition, convergence of $d\mu_{A_n}$ as distributions, on test functions which vanish at $\omega = 0$. \square

Energy spectra as functions of space momentum (pseudo-momentum in the case of lattice invariance) depend on commutators of the form $\langle [A^*(x,0)A(y,t)] \rangle$; for A hermitean and of definite parity, and $\tilde{f}(x) = \tilde{f}(-x)$, i.e. \tilde{f} real, parity (P) invariance of the dynamics and of Ω implies

$$\langle A(f,0) A(f,t)^* \rangle = \langle P A(f,0)^* A(f,t) P \rangle = \langle A(f,0)^* A(f,t) \rangle , \quad (10)$$

i.e. $d\mu_{A(f)^*}(\omega) = d\mu_{A(f)}(\omega)$, so that Lemma 1 applies and the Fourier transform

$$d\nu_{A(f)}(\omega) \equiv \mathcal{F}(-id/dt \langle [A(\bar{f},0), A(f,t)] \rangle)$$

is a positive even measure. For states invariant under space translations,

$$\langle [A(f,0)^*, A(f,t)] \rangle = \int \langle A (dE(k,\omega) - dE(k,-\omega)) A \rangle \tilde{f}(k)^2 e^{i\omega t} , \quad (11)$$

so that

$$d\nu_{A(f)}(\omega) = \omega(1 - e^{-\beta\omega}) \int \langle A dE(k,\omega) A \rangle \tilde{f}(k)^2 \quad (12)$$

for a KMS state, and similarly for a ground state. For $f_{k,R}(x) = e^{ikx} \alpha_R(x)$, α_R as in eq.(7), $R \rightarrow \infty$,

$$d\nu_{A,f_{k,R}}(\omega) \equiv 1/R^d d\nu_{A(f_{k,R})}(\omega) \rightarrow \omega(1 - e^{-\beta\omega}) \langle A dE(k,\omega) A \rangle \quad (13)$$

apart from a constant, as distributions in ω , assuming a polynomial bound in t for the L^1 norm of the commutator $\langle [A(0,0), A(y,t)] \rangle$ [9].

For states invariant under a lattice group of space translations, $\langle [A(x+a, 0)A(y+a, t)] \rangle$ is periodic in a and, for $f \equiv f_{k,R}$, apart from an irrelevant constant

$$\begin{aligned} & 1/R^d \langle [A(\bar{f}), A(f, t)] \rangle \rightarrow \\ & \int_{\Lambda \times \Lambda} dx dy \sum_n \langle [A(x, 0), A(y+n, t)] \rangle e^{ik(y-x)} e^{ikn} = \\ & = \int_{\Lambda \times \Lambda} dx dy \int \langle A(x, 0) (dE(k, \omega) - dE(k, -\omega)) A(y, 0) \rangle e^{ik(y-x)} e^{i\omega t} \end{aligned}$$

for $R \rightarrow \infty$, assuming integrability of the above commutator, with Λ denoting a lattice cell (the integrand being periodic both in x and y). Therefore, as distributions in ω ,

$$\begin{aligned} d\nu_{A, f_{k,R}}(\omega) & \rightarrow \text{const } \omega(1 - e^{-\beta\omega}) \times \\ & \int_{\Lambda \times \Lambda} dx dy \langle A(x, 0) dE(k, \omega) A(y, 0) \rangle e^{ik(y-x)} , \end{aligned} \quad (14)$$

assuming, as above, a polynomial bound in t for the commutators.

In the following, the second and fourth momenta of $d\nu_{A, f_{k,R}}$ and their limits for $R \rightarrow \infty$ will be calculated and discussed in terms of powers of k , for $A = \rho(h)$, $h(x) \geq 0$. The one point result for the energy spectrum of plasma waves at zero momentum will use the following

Lemma 2 *If a sequence of positive even measures $d\nu_n(\omega)$ satisfies*

$$\lim_n \int d\nu_n \omega^{2k} = c^{2k} \quad , \quad k = 0, 1, 2 \quad , \quad c > 0 \quad ,$$

then the sequence converges in the sense of measures to

$$1/2 (\delta(\omega - c) + \delta(\omega + c))$$

If $\lim_n \int d\nu_n = 1$ and $\lim_n \int d\nu_n \omega^2 = 0$, then it converges to $\delta(\omega)$.

Proof: $(\omega^2 - c^2)^2 d\nu_n(\omega)$ are finite positive measures, their integrals converge to zero for $n \rightarrow \infty$ and therefore, for any continuous bounded function f , $\int f(\omega)(\omega^2 - c^2)^2 d\nu_n(\omega)$ converge to zero. Any continuous, bounded, even function g can be approximated uniformly by

$$g(c) + f_\varepsilon(\omega)(\omega^2 - c^2)^2$$

with f_ε continuous and bounded; this implies

$$\int g(\omega) d\nu_n \rightarrow g(c)$$

and this is enough since the $d\nu_n$ are even. The second statement follows similarly. \square

3. Equations of motion and low k expansions

In the following, we analyze the energy spectrum associated to density waves in infinite fermion systems. As a consequence of eq.(9), the calculation of the momenta of the corresponding measures $d\nu_{\rho(h),f}(\omega)$ reduces to the application of the ACR relations to the time derivatives of ρ . By time translation invariance of Ω , in order to obtain the momenta up to the fourth (the fifth for $d\mu$), only three time derivatives must be computed.

Assumptions A), B), C) will be needed only for the Wick polynomials of second degree, the only ones which appear in the time derivatives of $\rho(x) \equiv \psi^*(x)\psi(x)$ (eqs.(14) and (19)-(24)). As usual, $:M:$ will denote the Wick ordered polynomial M in the variables $\psi^*(x_i), \psi(x_i)$, $i = 1 \dots N$ i.e the polynomial with all the ψ^* on the left of all the ψ and the sign of the corresponding permutation of the variables. For simplicity, we will omit the fermion field indexes.

Considerable information about the low momentum behaviour of charge density commutators can be obtained from general principles. The first observation is that each time derivative *with respect to the free evolution* of any second degree gauge invariant polynomial $P(x)$ in the fermion fields explicitly introduces one more space derivative in the expression of $P(x, t)$, and therefore one power of k in the Fourier transform of any correlation function of P .

Lemma 3 *For gauge invariant polynomials of second degree, $P(x) = \sum a_{n,k} \partial^{(n)} \psi^*(x) \partial^{(k)} \psi(x)$, $\partial^{(n)}$ denoting a product of n space derivatives, the commutator with the free Hamiltonian H_R^0 , in the limit $R \rightarrow \infty$, is of the form $\sum_i \partial_i Q_i(x)$, with Q_i in the same class.*

Proof: For the free evolution, from

$$d/dt \psi(x) = i \lim_R [H_R^0, \psi(x)] = i/2m \Delta \psi(x) \quad (15)$$

and the conjugate equation it follows immediately

$$\begin{aligned} d/dt \partial^{(n)} \psi^*(x) \partial^{(k)} \psi(x) = \\ = i/2m \partial_l (-\partial_l \partial^{(n)} \psi^*(x) \partial^{(k)} \psi(x) + \partial^{(n)} \psi^*(x) \partial_l \partial^{(k)} \psi(x)) \quad \square \end{aligned} \quad (16)$$

A second source of powers of k arises from symmetries which imply the vanishing of the integral of the commutators $[A(x), B]$, for all local variables B , as operator valued distributions. In fact, by locality of the ACR relations such commutators have compact support, their Fourier transform is analytic in k and vanishes at $k = 0$ if the above

property holds. E.g., since the current j_i generates space translations, the expectation value of current commutators on a space translation invariant state vanish at zero momentum:

$$\int dx \Omega([j_i(x), B]) = 0 \quad (17)$$

From parity invariance of Ω and definite parity of A , it also follows that

$$\langle A(x) d^n/dt^n A(y) \rangle$$

is even and therefore its Fourier transform, if regular enough, is of order $2m + 2$ in k if it vanishes of order $2m$.

A basic result for the following is obtained from the commutator between the current

$$j_i(x) = i/2m (\partial_i \psi^*(x) \psi(x) - \psi^*(x) \partial_i \psi(x))$$

and the interaction hamiltonian, which gives rise to the density of force

$$F_i(x) \equiv : \rho(x) \int dy \partial_i V_L(x - y) \rho(y) : \quad (18)$$

The commutator of $F_i(x)$ with any local variable is integrable, also in the limit $L \rightarrow \infty$ if $\partial_i V(x)$ is integrable; it vanishes after integration in x since it is odd in the interchange of x and y (a consequence of the third law of Newton's). The same holds for the total force, for any number of fields. In the case of non integrable potentials, a subtraction is needed and a convenient relation is given by the following

Lemma 4 *For a space translation invariant state, if*

$$\langle [: (\rho(x) - \langle \rho \rangle) \partial_i V(x - y) \rho(y) : , B] \rangle , \quad (19)$$

is integrable in the two variables, then its integral vanishes and

$$\int dx \lim_{L \rightarrow \infty} \int dy \langle [: (\rho(x) - \langle \rho \rangle) \partial_i V_L(x - y) \rho(y) : , B] \rangle = 0 . \quad (20)$$

In the absence of translation invariance, the same result holds with $\langle \rho \rangle$ replaced by $\langle \rho(x) \rangle$ and $\rho(y)$ replaced by $(\rho(y) - \langle \rho(y) \rangle)$

Proof: Using integrability, the l.h.s. of eq. (20) gives the integral of (19), by the Lebesgue dominated convergence theorem. On the other side, for all L ,

$$\int dy \partial_i V_L(x - y) \langle \rho \rangle = 0$$

and therefore the l.h.s. of eq.(20) is independent of the substitution of $\rho(y)$ with $\rho(y) - \langle \rho \rangle$; in this form, the integral vanishes for all L by

antisymmetry in the exchange of x and y . The last statement follows from integrability and antisymmetry. \square

In the application below, the subtraction $\rho(x) - \langle \rho \rangle$ in Lemma 4 is a priori irrelevant for $\partial_i V(x)$ integrable (and in fact also for $\partial_i \partial_j V(x)$ integrable), but crucial for the Coulomb potential, where actually only the subtracted terms will survive in the analysis of the frequency spectrum in the zero momentum limit.

The equations of motion for the density are the following

$$\frac{d}{dt}\rho(x, t) = -\partial_i j_i(x) \quad (21)$$

$$\frac{d^2}{dt^2}\rho(x, t) = 1/m \partial_i (\partial_k S_{ki}(x) + : \rho(x) \partial_i (V_L * \rho)(x) :) \quad (22)$$

$$\begin{aligned} \frac{d^3}{dt^3}\rho(x, t) = & +1/m \partial_i \partial_k (\dot{S}_{ki}^0(x) + \dot{S}_{ki}^V(x)) + \\ & - 1/m \partial_i (: \partial_k j_k(x) \partial_i (V_L * \rho)(x) : + : \rho(x) \partial_i (V_L * \partial_k j_k(x) :)) \end{aligned} \quad (23)$$

with

$$S_{ki}(x) = 1/4m (\partial_k \psi^* \partial_i \psi - \partial_k \partial_i \psi^* \psi + h.c.) \quad , \quad \dot{S}_{ki}^0 = \partial_l S_{lki} \quad , \quad (24)$$

S_{lki} being obtained as in Lemma 3 and

$$\dot{S}_{ki}^V(x) = - (: j_k(x) \partial_i V_L * \rho(x) : + (i \leftrightarrow k)) \quad (25)$$

In the presence of spin, exactly the same equations hold for ρ, j_i and S_{ij} replaced by the corresponding sum over the spin index. For N kinds of fermion fields, ψ_l , $l = 1 \dots N$ with masses m_l , the same equations apply to the density operators $\rho^{(l)}$, the corresponding currents $j^{(l)}$ and stress tensors $S_{ik}^{(l)}$, with mass m_l and the obvious substitution of $\partial_i V_L * \rho$ with $\sum_m \partial_i V_L^{(l,m)} * \rho^{(m)}$. Since the potential is assumed to be regularized by the convolution with $\gamma(x) \in \mathcal{S}$, and the correlation functions of $\rho(x)$, $j_i(x)$, $S_{ij}(x)$ are assumed to be bounded after smearing, the space cutoff L can be removed in all correlation functions, for all potentials with integrable derivatives.

Wick ordering in eqs.(22), (23), (25) can be omitted if the partial derivatives of the potential vanish at the origin up to the third order. Since this leads to simplifications in the calculations (only a few commutators are then needed, and one can forget about Fermi fields and their ordering) with no substantial consequences on the results [28], such a property will be assumed in the following for the regularization γ of the potentials (the derivatives of V_L vanishing at the origin up

to the third order if $\Delta\gamma(0) = 0$). Our calculations will only use the above time derivatives and the following basic commutators:

$$[\rho(y), j_j(z)] = (-i)/m \partial_j^y (\delta(y-z) \rho(y)) \quad (26)$$

$$[\rho(y), S_{ik}(z)] = -i (\partial_i^y (\delta(y-z) j_k(y)) + (i \leftrightarrow k)) \quad (27)$$

$$[j_j(y), S_{ik}(z)] = -i/m (\partial_k^y \delta(y-z) S_{ij}(z) + (i \leftrightarrow k)) + \\ + i/m \partial_j^z (\delta(y-z) S_{ki}(z)) - i/4m^2 \partial_i^y \partial_k^y \partial_j^z (\delta(y-z) \rho(y)) \quad (28)$$

Eqs.(26),(27),(28) also hold for the sum over the spin index of the same operators. By eq.(9), the required momenta are of the form

$$\int \omega^n d\nu_{\rho(h), f_{k,R}} = \\ = 1/R^d (-i)^{n+1} \langle [\rho(h * \bar{f}_{k,R}), d^{n+1}/dt^{n+1} \rho(h * f_{k,R}, t)] \rangle |_{t=0} \quad (29)$$

For translation invariant states,

$$\int \omega^n d\nu(\omega) = (2\pi)^{-d} \int dq \tilde{T}_n(q) \tilde{h}(q)^2 \tilde{\alpha}_R(q-k)^2 / R^d \quad (30)$$

with

$$\tilde{T}_n(k) \equiv \mathcal{F} T_n(x-y) \equiv (-i)^{n+1} \mathcal{F} \langle [\rho(x), d^{n+1}/dt^{n+1} \rho(y, t)] \rangle |_{t=0} \quad (31)$$

From the equations of motion and boundedness of the (equal time) correlation functions after smearing it follows immediately that $\tilde{T}_n(k)$ is regular (C^∞ for potentials of fast decrease, continuous for potentials with $\partial_i V(x)$ integrable), so that the limit $R \rightarrow \infty$ of eq.(30) exists and gives $\tilde{T}_n(k)$, apart from a constant times $\tilde{h}(k)^2$, which is positive for small k . The corresponding normalized momenta of the frequency converge therefore to

$$< \omega^n(k) > = \tilde{T}_n(k) / \tilde{T}_0(k) \quad (32)$$

For states invariant under lattice translations (omitting for simplicity the convolution with h),

$$\int \omega^n d\nu_{\rho(0), f_{k,R}}(\omega) = \\ = 1/R^d (-i)^{n+1} \int dx dy e^{ik(y-x)} \alpha_R(x) \alpha_R(y) T_n(x, y) \quad (33)$$

with $T_n(x, y)$ given by the r.h.s. of eq.(31). For potentials with $\partial_i V(x)$ integrable, $T_n(x, y)$ is integrable in y and its integral is periodic in x .

Therefore, the limit $R \rightarrow \infty$ of eq.(33) exists and gives, apart from an irrelevant constant,

$$1/|\Lambda| \int_{\Lambda} dx \int dy e^{ik(y-x)} T_n(x, y) \equiv \tilde{T}_n(k, 0) \quad , \quad (34)$$

with the integrand periodic in x , the notation referring to the Fourier series expansion in the variable x . Again, with the notation of eq.(34)

$$\langle \omega^n(k) \rangle = \tilde{T}_n(k, 0) / \tilde{T}_0(k, 0) \quad (35)$$

Eq.(26) gives

$$\tilde{T}_0(k) = \langle \rho \rangle k^2 / m \quad (36)$$

$$\tilde{T}_0(k, 0) = 1/|\Lambda| \int_{\Lambda} dx \langle \rho(x) \rangle k^2 / m \equiv \langle \bar{\rho} \rangle k^2 / m \quad (37)$$

Eqs.(21),(22) immediately imply that $\tilde{T}_2(k)$ and $\tilde{T}_2(k, 0)$ are infinitesimal with respect to k^2 , as a consequence of the presence of space derivatives (also following from Lemma 3) and of the arguments following eq.(18). This also clearly applies for more than one kind of fermions, for the total density $\rho^{tot}(x) \equiv \sum_l \rho^{(l)}(x)$. Lemma 2 then gives a result similar to the Goldstone theorem for the spontaneous breaking of the Galilei group [27], [9]:

Proposition 1 *For any number of fermion fields and potentials $V^{(lm)}$ with integrable first derivatives, the energy spectral measures $d\nu_{\rho^{tot}(h), f_{k,R}}$, eqs.(13),(14), normalized to total unit mass, converge as measures, for $R \rightarrow \infty$ and $k \rightarrow 0$, to $\delta(\omega)$.*

In the next Section the mean squared frequency is discussed for all potentials decaying at infinity faster than the Coulomb case; as a result, Proposition 1 will be extended to all such potentials, more precisely to *all potentials with integrable second derivatives*

4. $\langle \omega^2(k) \rangle$ and long range potentials

The calculation of $\tilde{T}_2(k)$ and $\tilde{T}_2(k, 0)$ only requires eqs.(21), (22), (26), (28). In fact, using time translation invariance of Ω ,

$$\begin{aligned} T_2(x, y) &= -(-i)^3 \langle [d/dt \rho(x, t), d^2/dt^2 \rho(y, t)] \rangle|_{t=0} = \\ &= (-i)^3 / m \partial_i^x \partial_j^y (\partial_k^y \langle [j_j(x), S_{ki}(y)] \rangle + \langle [j_j(x), \rho(y) \partial_i^y (V_L * \rho)(y)] \rangle) = \\ &= 1/m^2 \partial_i^x \partial_j^y \partial_k^y (\partial_k^x \delta(x-y) S_{ij}(y) + \dots + 1/4 \partial_i^x \partial_k^x \partial_j^y \delta(x-y) \rho(x)) + \\ &\quad + 1/m^2 \partial_i^x \partial_j^y \int dz \partial_i \partial_j V_L(y-z) (\delta(x-y) - \delta(x-z)) \langle \rho(y) \rho(z) \rangle + \end{aligned}$$

$$+ 1/m^2 \partial_i^x \Delta^y (\delta(x-y) \langle \rho(y) \partial_i V_L * \rho(y) \rangle) \quad (38)$$

The dots refer to two terms obtained, as in eq.(28) by permutations of indexes. In the presence of spin, each ρ and S_{ij} operator must be summed over the spin index. The cutoff L can be removed if $\partial_i \partial_j V(x)$ is integrable, as a consequence of boundedness (after smearing) of the two point correlation function of ρ .

For translation invariant states, the last term in eq.(38) vanishes by invariance under parity (or, see eq.(22), under space and time translations), so that

$$\tilde{T}_2(k) = 1/m^2 k_i k_k (3 \langle S_{ik} \rangle k^2 - (\tilde{G}_{ik}(k) - \tilde{G}_{ik}(0))) + 1/4m^3 \langle \rho \rangle k^6 \quad (39)$$

with

$$\tilde{G}_{ik}(k) \equiv \mathcal{F}(\partial_i \partial_k V(x-y) \langle \rho(x) \rho(y) \rangle)$$

Eqs.(30), (36), (39) give, for the second momentum of the frequency

$$\langle \omega^2(k) \rangle = \frac{3 \langle S_{ik} \rangle}{m \langle \rho \rangle} k_i k_k + \frac{1}{4m^2} k^4 - \frac{k_i k_k}{m \langle \rho \rangle k^2} (\tilde{G}_{ik}(k) - \tilde{G}_{ik}(0)) \quad (40)$$

In the rotation invariant case,

$$\langle S_{ik}(x) \rangle = 1/dm \langle \partial_j \psi^* \partial_j \psi \rangle \equiv 2/d E_{kin} \langle \rho \rangle$$

Eq.(40) has been derived in refs. [11], [29] in the framework of a finite particle system in a large box (the use of the Coulomb potential in its derivation is not without problems, the plasma frequency being given by a singularity at the origin in $G_{ij}(k)$). It holds, in our framework, for all potentials with $\partial_i \partial_j V$ integrable.

For states invariant under a lattice subgroup of translations, the last term in eq.(38) vanishes by parity invariance after integration in x on a lattice cell; using periodicity in x and integration by parts,

$$\tilde{T}_2(k, 0) = 1/m^2 k_i k_k (3 \langle \bar{S}_{ik} \rangle k^2 - (\tilde{G}_{ik}^0(k) - \tilde{G}_{ik}^0(0))) + 1/4m^3 \langle \bar{\rho} \rangle k^6, \quad (41)$$

with $\bar{\rho}$ and \bar{S}_{ik} mean values over a lattice cell and

$$\tilde{G}_{ik}^0(k) \equiv 1/|\Lambda| \int_{\Lambda} dx \int dy e^{ik(y-x)} \partial_i \partial_k V(x-y) \langle \rho(x) \rho(y) \rangle,$$

so that, by eqs.(35),(37), the mean squared frequency is given again by eq.(40), with ρ and S_{ik} substituted by their mean values and \tilde{G}_{ik} by \tilde{G}_{ik}^0 .

In all the cases, the low momentum behaviour of the mean square frequency depends on the decay property the potential at infinity. In fact, $\tilde{G}_{ij}(k)$ and $\tilde{G}_{ij}^0(k)$ are C^n for potentials with $|x|^n \partial_i \partial_j V$ integrable.

For $n = 0$, the (immediate) extension of eqs.(40),(41) to N kinds of fermions proves Proposition 1 for potentials with integrable second derivatives. Actually, the result only depends on an additional power of the momentum appearing in all the commutators as a consequence of eq.(17). For $n \geq 4$, in particular for sufficiently regular potentials decaying at least as $|x|^{-d-2-\varepsilon}$, eq.(40) implies, using rotation invariance,

$$\langle \omega^2(k) \rangle = (6/d E_{kin} + C) k^2 / m \langle \rho \rangle + O(k^4) \quad (42)$$

The same holds for $n \geq 2$ (in particular for regular potentials decaying at least as $|x|^{-d-\varepsilon}$), with $O(k^4)$ replaced by $o(k^2)$. In both cases C can be written, in dimension $d = 3$,

$$C = 1/30 \int d^3x \langle \rho(0) \rho(x) \rangle (x^2 \Delta + 2x_i x_j \partial_i \partial_j) V(x)$$

For potentials with $|x|^\beta \partial_i \partial_j V(x)$ integrable, $0 < \beta < 2$, the last term in eq.(40) can dominate at low momentum; it can be estimated as:

$$|\tilde{G}_{ij}(k) - \tilde{G}_{ij}(0)| \leq 2 |k|^\beta \int dx |x|^\beta |G_{ij}(x)| \quad ,$$

so that

$$\langle \omega^2(k) \rangle \leq \text{const } |k|^\beta \quad . \quad (43)$$

If the potential term dominates, the low momentum behaviour of $\langle \omega^2(k) \rangle$ is clearly independent from the addition of any potential W with $x^2 \partial_i \partial_j W$ integrable; for potentials $V(x) \sim |x|^{-\alpha}$, $d-2 < \alpha < d$, one obtains

$$\langle \omega^2(k) \rangle \sim |k|^\beta$$

with $\beta = \alpha - d + 2$. The same results apply to states invariant under lattice translations.

In the following Sections the fourth momentum of the frequency is discussed, in the zero momentum limit for the Coulomb interaction and to the order k^2 for short range potentials.

5. The plasmon spectrum in Jellium

We will derive in this Section the one point result, eq.(1), for the energy spectrum of density waves in the limit $k \rightarrow 0$ for the jellium model, in three space dimensions; the result will follow from the application of Lemma 4 to the analysis of the fourth momentum of the frequency and of Lemma 2 to the momenta up to the fourth.

As discussed above, in the case of a Coulomb potential an infrared cutoff is necessary in the equations of motion. Its removal in the first

four momenta of the frequency only requires that the two and three point correlation functions of $\rho(x) - \langle \rho(x) \rangle$ decay (respectively) as $|x_1 - x_2|^{-1-\varepsilon}$ and $|x_1 - x_2|^{-1-\varepsilon} |x_2 - x_3|^{-1-\varepsilon}$.

Eqs.(40),(41) for the second momentum of the frequency holds for potentials V_L with ultraviolet and infrared regularization, eqs.(6),(7). The first two terms in eq.(40), are of order k^2 and k^4 ; for the last term, from integrability in y of eq.

$$\partial_j V(x-y)(\langle \rho(x)\rho(y) \rangle - \langle \rho(x) \rangle \langle \rho(y) \rangle)$$

it follows that, after such a subtraction, the limit for $L \rightarrow \infty$ vanishes of order k . We are left therefore with

$$\mathcal{F}(\partial_i \partial_j V_L(x-y) \langle \rho(x) \rangle \langle \rho(y) \rangle)$$

and therefore, in the space translation invariant case,

$$< \omega^2(k) > = \lim_L k^2 \tilde{V}_L(k) \langle \rho \rangle / m + O(k) \rightarrow e^2 \langle \rho \rangle / m \quad (44)$$

for $k \rightarrow 0$, using $\tilde{\gamma}(0) = 1$. $O(k)$ becomes $O(k^2)$ if the truncated correlation function of ρ decays as $|x|^{-2-\varepsilon}$.

For lattice translation invariant states, expanding the periodic function $\langle \rho(x) \rangle$ in Fourier series, the corresponding term reads

$$\frac{k_i k_j}{m \langle \bar{\rho} \rangle k^2} \sum_n |\langle \bar{\rho}_n \rangle|^2 ((k_i + n_i)(k_j + n_j) \tilde{V}_L(k+n) - n_i n_j \tilde{V}_L(n)) \quad ,$$

n ranging over the reciprocal lattice of the translation lattice. For $L \rightarrow \infty$, all the terms with $n \neq 0$ are of order k and the same holds for their sum (as a consequence of the regularization of the potential). Therefore $< \omega^2(k) >$ converges, for $k \rightarrow 0$, to $e^2 \langle \bar{\rho} \rangle / m$ (in fact $\bar{\rho} = \tilde{\rho}_0$), i.e. to the square of the plasma frequency associated to the mean density.

The absence of two powers of k in eq.(44) with respect to eq. (42) depends on the failure, for the Coulomb potential, of both the mechanisms mentioned in Sect.3, i.e., in the subtraction needed for the validity of Newton's third law and in the non-integrability in x of the commutator

$$-i \int dy \langle [j_i(x), \partial_k V(y) \rho(y)] \rangle = -1/m \partial_k \partial_i V(x) \langle \rho(x) \rangle \quad ,$$

which in fact gives, when summed over equal indexes, $e^2 \langle \rho \rangle$ (Gauss' law). The result for the mean squared frequency also follows from an analysis of the removal of the infrared cutoff in the above commutator:

$$-1/m \int dx \partial_i \partial_i V_L(x) \rho(x) = e^2/m \int dx (\delta(x) - \sigma_L(x)) \rho(x)$$

with $\sigma_L(x)$ of support near $|x| = L$ and $\int \sigma_L(x) dx = 1$, so that, in all correlation functions, for $L \rightarrow \infty$, $\int dx \sigma_L(x) \rho(x) \rightarrow \langle \bar{\rho} \rangle$

The fourth momentum of the frequency is obtained, eqs.(31),(32), from the commutator between the r.h.s. of eqs.(22),(23). The complete commutator has been calculated in ref. [28]. In order to discuss the zero momentum limit, by eqs.(32),(36),(37), we need only terms up to the second order in k , i.e. the commutators between the last terms in equations (22), (23). We need therefore to discuss the limit $L \rightarrow \infty$ and then $k \rightarrow 0$ of

$$\langle [\rho(x) \partial_i V_L * \rho(x), \partial_k j_k(y) \partial_j V_L * \rho(y) + \rho(y) \partial_j V_L * \partial_k j_k(y)] \rangle \quad (45)$$

By applying eq.(26), four terms appear, each involving the three point function of the density. In the translation invariant case, y can be fixed and $\langle \rho \rangle$ subtracted in the convolutions. If we also subtract to the l.h.s. of the commutator a term $\langle \rho \rangle \partial_i V_L * \rho$, such terms are integrals, in two variables, of functions bounded by integrable functions uniformly in L , as a consequence of the decay assumptions on the correlation functions of the density. Their limit $L \rightarrow \infty$ vanishes by Lemma 4 and therefore we can substitute $\langle \rho \rangle$ to $\rho(x)$ in the l.h.s. of the commutator. The same argument then applies, by antisymmetry of $\partial_i V(z)$ to the r.h.s., for the subtraction $\rho(y) \rightarrow \rho(y) - \langle \rho \rangle$. We are therefore left with

$$\partial_i^x \partial_j^y \langle [\langle \rho \rangle \partial_i V_L * \rho(x), \langle \rho \rangle \partial_j V_L * \partial_k j_k(y)] \rangle$$

and eq.(26) gives

$$\begin{aligned} \langle \rho \rangle^2 \int dw dz \triangle V_L(x-w) \triangle V_L(y-z) (-i)/m \partial_k^z \partial_k^w \langle \delta(w-z) \rho(w) \rangle = \\ = -i/m \langle \rho \rangle^3 \int dz \partial_k \triangle V_L(x-z) \partial_k \triangle V_L(y-z) , \end{aligned}$$

so that, by eq.(31),

$$\mathcal{F} T_4(k) \sim 1/m^3 \langle \rho \rangle^3 (k^2)^3 \tilde{V}_L^2(k) \rightarrow 1/m^3 \langle \rho \rangle^3 e^4 k^2 \tilde{\gamma}^2(k)$$

for $L \rightarrow \infty$. This immediately implies

$$\langle \omega^4(k) \rangle = \tilde{T}_4(k)/\tilde{T}_0(k) \rightarrow e^4 \langle \rho \rangle^2 / m^2 \quad (46)$$

for $k \rightarrow 0$, independently of the regularization γ , with $\tilde{\gamma}(0) = 1$. By Lemma 2, eqs. (44), (46) imply the one point result, eq.(1), for the energy spectrum of density waves in the zero momentum limit. The result is independent from the ultraviolet regularization of the potential and clearly holds for all potentials with $k^2 \tilde{V}(k) \rightarrow e^2$ for $k \rightarrow 0$.

For states invariant under lattice translations, eqs.(34),(35) apply and we must consider the mean in x and integral in y of the commutator (45). Using the cluster properties and applying Lemma 4, the commutator vanishes, for $L \rightarrow \infty$, at zero momenta, after the subtractions $\rho \rightarrow \rho - \langle \rho(x) \rangle$ in the l.h.s; the remaining terms vanish, as above, after the same subtraction in the r.h.s., so that we end with the expectation value of the commutator between

$$\langle \rho(x) \rangle (\partial_i V_L * \rho)(x) + \rho(x) \langle (\partial_i V_L * \rho)(x) \rangle$$

and

$$\partial_k j_k(y) \langle (\partial_j V_L * \rho)(y) \rangle + \langle \rho(y) \rangle \partial_j V_L * \partial_k j_k(y)$$

Using integration by parts in eq.(34) and discarding terms of order $> k^2$ (uniformly in L), eq.(26) gives

$$\tilde{T}_4(k, 0) \sim \lim_L 1/m^3 |\Lambda| \int_{\Lambda} dx \int dy dw dz \partial_i \partial_k V_L(x-w) \partial_j \partial_k V_L(y-z)$$

$$k_i k_j e^{ik(y-x)} \rho_x \rho_w (\rho_y (\delta(w-z) - \delta(x-z)) + \rho_z (\delta(x-y) - \delta(y-w))) \quad (47)$$

with $\rho_a \equiv \langle \rho(a) \rangle$. In the translation invariant case, only the first term appeared in the r.h.s., the others corresponding to an irrelevant subtraction of a constant in the convolutions. By the previous result on the second momentum of the frequency, the one point result for the plasma spectrum is equivalent to the cancellation of all the terms different from the mean in the Fourier expansion of the expectation value of the density. The expansion gives, with $\tilde{\rho}_n$ the Fourier coefficients of ρ_x ,

$$\lim_L \sum_{n_1+n_2+n_3=0} \tilde{\rho}_{n_1} \tilde{\rho}_{n_2} \tilde{\rho}_{n_3} (\partial_i \partial_k \tilde{V}_L(-k-n_1) - \partial_i \partial_k \tilde{V}_L(n_2))$$

$$(\partial_j \partial_k \tilde{V}_L(k-n_3) - \partial_j \partial_k \tilde{V}_L(n_3)) \quad ;$$

by the regularity of $\tilde{V}(k)$ for $k \neq 0$ and the vanishing of $\partial_j \partial_k \tilde{V}_L(0)$, all the terms are of order k uniformly in L , except those with $n_3 = 0$; in this case, since $n_1 = -n_2$, the limit is non vanishing only for $n_1 = n_2 = n_3 = 0$. By the fast decrease of $\tilde{V}(k)$ the same holds for the sum. The result is therefore, at the order k^2

$$\tilde{T}_4(k, 0) \sim \langle \bar{\rho} \rangle^3 / m^3 (k^2)^3 \tilde{V}(-k) \tilde{V}(k) \sim e^4 \langle \bar{\rho} \rangle^3 k^2 / m^3 \quad ; \quad (48)$$

as before, eq.(48) holds independently of the ultraviolet regularization γ , only requiring $k^2 \tilde{V}(k) \rightarrow e^2$ for $k \rightarrow 0$, and implies eq.(46), with $\langle \rho \rangle$ replaced by the mean density $\langle \bar{\rho} \rangle$. In the presence of spin, the same derivation and results apply to the charge and current operators summed over the spin index.

6. $\langle \omega^4(k) \rangle$ at the order k^2

We calculate the fourth momentum of the frequency at the order k^2 , for translation and rotation invariant states, from eqs.(31),(32). From Sect.3 it follows that, for potentials of fast decrease, $T_4(k)$ is regular (C^∞) and at least of order k^4 . In ref.[28] $T_4(k)$ has been obtained, at the order k^4 , in terms of two point functions $\langle \rho S_{ik} \rangle$, $\langle j_i j_k \rangle$ and of the three point function of ρ . Using identities which follow from the equation of motion, we shall express $T_4(k)$ at the order k^4 , and therefore $\langle \omega^4(k) \rangle$ at the order k^2 , purely in terms of the three point function of the density, for one component models with short range potentials. By eq.(31),

$$T_4(x-y) = (-i) \langle [d^2/dt^2 \rho(x,t), d^3/dt^3 \rho(y,t)] \rangle \quad (49)$$

at $t = 0$. In the commutator of the r.h.s. of eqs.(22),(23) we drop $\dot{S}_{ik}^0(x)$, which produces terms of order k^6 (also as a consequence of the cancellations discussed in Sect.3), use the identity (for A and B of the same definite parity)

$$\langle [A(x), \dot{B}(y)] \rangle = \langle [B(y), \dot{A}(x)] \rangle = \langle [B(x), \dot{A}(y)] \rangle$$

and eq.(25). We obtain, using translation invariance,

$$\begin{aligned} im^2 T_4(x-y) = & -2 \partial_i \partial_k \partial_j \partial_m \langle [S_{ki}(x), j_j(y) \partial_m V * \rho(y)] \rangle \\ & -4 \partial_i \partial_j \partial_m \langle [\rho(x) \partial_i (V * \rho)(x), j_j(y) \partial_m V * \rho(y)] \rangle \\ & + \partial_i \partial_j \langle [\rho(x) \partial_i (V * \rho)(x), (\partial_k j_k(y) \partial_j (V * \rho)(y) + \rho(x) \partial_j (V * \partial_k j_k(y))] \rangle \end{aligned} \quad (50)$$

where ∂ denotes the derivation with respect to x . The second and third commutators only require eq.(26), so that the result involves purely the three point function of ρ . The first commutator requires eqs.(28) and (27); dropping from eq.(28) the last two terms, of higher order in k , omitting the four derivatives and integrating in y we obtain, for the coefficient of $k_i k_j k_k k_m$,

$$4i \int dz \partial_k \partial_m V(x-z) 1/m \langle S_{ij}(x) \rho(z) - j_i(x) j_j(z) \rangle$$

Rotation invariance, symmetry under the permutations $i \leftrightarrow j$ and $i \leftrightarrow k$ and summation over equal indices give, for the coefficient of k^4 ,

$$4i/5 \int dz V(x-z) (1/m \langle \partial_i \partial_j S_{ij}(x) \rho(z) \rangle + \langle \partial_i j_i(x) \partial_j j_j(z) \rangle) .$$

The identity

$$0 = d/dt \langle \dot{\rho}(x,t) \rho(z,t) \rangle|_{t=0} = \langle \partial_i j_i(x) \partial_j j_j(z) \rangle +$$

$$+ 1/m \langle \partial_i \partial_j S_{ij}(x) + \partial_i(\rho(x) \partial_i V * \rho(x)) \rho(z) \rangle$$

then gives, for the first term in T_4 ,

$$\tilde{T}_4^{(1)}(k) = -4/5 k^4/m^3 \int dy dz \partial_i V(y) \partial_i V(z) \langle \rho(0) \rho(y) \rho(z) \rangle \quad (51)$$

The second and third commutators in eq.(50) are immediately calculated from eq.(26); in both commutators, a term with four space derivatives appears, of the same form as in eq.(51), with coefficients, respectively, $4/3$ and $1/3$. The remaining terms only contain three or two space derivatives, and a Taylor expansion in k of the corresponding expressions is necessary; the resulting contributions to $T_4(z)$ are therefore integrals of the three point function of the density with first and second order polynomials in z . Summing all the terms, we obtain, to the fourth order in k ,

$$\begin{aligned} \tilde{T}_4(k) = & (-4/5 + 5/3) k^4/m^3 \int dy dz \partial_i V(y) \partial_i V(z) \langle \rho(0) \rho(y) \rho(z) \rangle + \\ & + k_i k_j k_l k_m / m^3 \int dy dz Z_{ijlm}(y, z) \langle \rho(0) \rho(y) \rho(z) \rangle, \end{aligned} \quad (52)$$

with

$$\begin{aligned} Z_{ijlm}(y, z) \equiv & 6 \partial_i V(y) z_m \partial_j \partial_l V(z) + \\ & + 1/2 \partial_i \partial_k V(y) z_l z_m \partial_j \partial_k (2V(z) - V(z - y)) \end{aligned} \quad .$$

By rotation invariance, the result can also be written as

$$\tilde{T}_4(k) = k^4/m^3 \int dy dz W(y, z) \langle \rho(0) \rho(y) \rho(z) \rangle \quad (53)$$

with

$$\begin{aligned} W(y, z) = & 13/15 \partial_i V(y) \partial_i V(z) + 6/15 \partial_i V(y) (z_i \Delta + 2z \cdot \partial \partial_i) V(z) + \\ & + 1/15 \partial_i \partial_k V(y) \partial_j \partial_k V(z) (\delta_{ij} y \cdot z + y_i z_j + y_j z_i) \end{aligned} \quad (54)$$

and eq.(2) follows.

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References

- [1] L. Tonks and I. Langmuir. Oscillations in ionized gases. *Phys. Rev.*, 33:195, 1929.
- [2] T. Nagamiya R. Kubo. *Solid State Physics*. McGraw Hill, 1969.
- [3] D. Pines. *Elementary excitations in solids*. Benjamin, N.Y., 1964.
- [4] R. P. Feynman. Atomic theory of the two-fluid model of liquid helium. *Phys. Rev.*, 94:262, 1954.
- [5] R.P.Feynman. in: *Progress in low temperature physics, vol.I, Cap. 2*. C.J.Gorter Ed. , North-Holland, Amsterdam, 1955.
- [6] D. Pines and P. Nozieres. *Theory of quantum liquids*. Benjamin, N.Y., 1966.
- [7] L.D.Landau. Theory of superfluidity of he ii. *J. Phys. USSR*, 5:71, 1941.
- [8] F. Strocchi. *Elements of quantum mechanics of infinite systems*. World Scientific, 1985.
- [9] G. Morchio and F. Strocchi. Spontaneous breaking of the galilei group and the plasmon energy gap. *Annals of Physics*, 170:310, 1986.
- [10] M. Broidioi and A. Verbeure. The plasmon in the one component plasma. *Helv. Phys. Acta*, 66:155, 1993.
- [11] R. D. Puff. Application of sum rules to the low-temperature interacting boson system. *Phys. Rev.*, 137:A406, 1965.
- [12] S. Stringari. Sum rules for density and particle excitations in bose superfluids. *Phys. Rev. B*, 46:2974, 1992.
- [13] S. Stringari. Spin excitations and sum rules in the heisenberg antiferromagnet. *Phys. Rev. B*, 49:6710, 1994.
- [14] D.M. Ceperley and B.J. Alder. Ground state of the electron gas by a stochastic method. *Phys. Rev. Lett.*, 45:566, 1980.
- [15] S.Ciccariello. Strongly localized quantum crystalline states of the jellium model. *arXiv*, cond-math:0712.1463v1, 2007.
- [16] F. Family. Sum rules and high-frequency behavior of dynamic structure function of quantum fluids. *Phys. Rev. Lett.*, 34:1374, 1975.
- [17] D. Forster, P. C. Martin, and S.Yip. Moments of the momentum density correlation functions in simple liquids. *Phys. Rev.*, 170:155, 1968.

- [18] J.P. Solovej E.H. Lieb, R. Seiringer. Ground-state energy of low-density fermi gas. *Phys. Rev. A*, 71:53605–1, 2005.
- [19] V. Rivasseau M. Disertori. A rigorous proof of fermi liquid behavior for jellium two-dimensional interacting fermions. *Phys. Rev. Lett.*, 85:361, 2000.
- [20] T.R.Kirkpatrick D.Belitz. Theory of many-fermion systems. *Phys. Rev.*, B 56:6513, 1997.
- [21] T.R.Kirkpatrick D.Belitz. Theory of many-fermion systems ii: The case of coulomb interactions. *Phys. Rev.*, B 58:9710, 1998.
- [22] O. Bratteli and D. Robinson. *Operators algebras and statistical mechanics 2*. Springer, 1981.
- [23] G. L. Sewell. *Quantum theory of collective phenomena*. Oxford Clarendon Press, 1986.
- [24] R.F. Streater and A. Wightman. *PCT, spin statistics and all that*. Addison-Wesley, 1964.
- [25] D. W. Robinson. Statistical mechanics of quantum spin systems. ii. *Comm. Math. Phys.*, 7:337, 1968.
- [26] G. Morchio and F. Strocchi. Spontaneous symmetry breaking and energy gap generated by variables at infinity. *Comm. Math. Phys.*, 99:153, 1985.
- [27] J. Swieca. Range of forces and broken symmetries in many-body systems. *Comm. Math. Phys.*, 4:1, 1967.
- [28] A. Cintio. *Regole di somma per sistemi infiniti*. Tesi, Dipart. di Fisica, Univ. di Pisa, 2002.
- [29] N. Iwamoto. Inequalities for frequency-moment sum rules of electron liquids. *Phys. Rev. A*, 33:1940, 1986.